Announcements

1) Change to office hours:

Today 2-3 instead of 1-2

2) Assignment 7 due Friday, supplement on CTools

Chapter 5

Eigenvalues and Eigenvectors

"Eigen" = inherent

If A is a diagonal matrix $(n \times n)$, then $\det(A)$ is the product of the diagonal entries.

Wishful Thinking:

Wouldn't it be nice if every $n \times n$ matrix were diagonal?

Definition: (eigenvalues/eigenvectors)

Let A be an $n \times n$ matrix and let λ be a scalar.

Then λ is an eigenvalue for A if there is a nonzero vector x with

$$\boxed{Ax = \lambda x}$$

The vector x is called an eigenvector associated to the eigenvalue λ.

What do they give you?

The largest eigenvalue of A (in absolute value) gives you "the maximum amount A can stretch any vector".

Example 1: (diagonal)

$$A = \begin{bmatrix} 16 & 0 \\ 0 & -132 \end{bmatrix}$$

Then you can check that

with $\lambda_1 = 16$, $\lambda_2 = -132$,

$$Ae_1 = \lambda_1 e_1 \quad \text{and}$$

$$Ae_2 = \lambda_2 e_2$$

$$\left(e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

In general: the eigenvalues of a diagonal matrix are just the diagonal entries - sometimes you may not get distinct eigenvalues, i.e.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Q: How to find in general?

A: The characteristic polynomial!

Suppose $Ax = \lambda x$ for A a matrix, x a vector, and λ a scalar. We can write this as $Ax = \lambda I_n x$.

Subtracting $\lambda I_n x$ from both sides, we get

$$Ax - \lambda I_n x = \vec{0}.$$

We can rewrite this as

$$\boxed{(A - \lambda I_n)x = \vec{0}}$$

If x is nonzero, $A - \lambda I_n$ is not invertible.

Why not? If it were, multiply both sides by $(A-\lambda I_n)^{-1}$ to get

$$x = (A-\lambda I_n)^{-1}\underbrace{(A-\lambda_n I_n)x}_{=}$$

$$= (A-\lambda I_n)^{-1}\vec{0}$$

$$= \vec{0}, \quad \text{so } x = \vec{0}.$$

But we said $x \neq \vec{0}$! So $A-\lambda I_n$ can't be invertible.

So to find the eigenvalues of A,

Compute

$$\boxed{\det(A - \lambda I_n) = 0}$$

and find all solutions λ.

Example 2: (2×2)

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

Find the eigenvalues of A.

Compute

$$\det\left(A - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 3 & 5-\lambda \end{bmatrix}\right)$$

$$= \lambda^2 - 6\lambda - 1 = 0$$

Use the quadratic formula:

$$\lambda = \frac{6 \pm \sqrt{36+4}}{2}$$

$$= \frac{6 \pm 2\sqrt{10}}{2}$$

$$= \boxed{3 \pm \sqrt{10}}$$

– or – you write

"eigenvalues of $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$"

in Wolfram Alpha.

This works for any

$n \times n$ matrix provided

you actually could

compute the eigenvalues!

If A is

2×2: quadratic formula

3×3: there is a formula for the roots of any cubic polynomial

4×4: there is a formula for the roots of any quartic polynomial

$n\times n$ where $n\geq 5$: no formula can possibly exist!

Even simpler problem:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\det\left(A - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right)$$

$$= \lambda^2 + 1 = 0$$

The solutions are not real numbers!

Example 3: (3×3)

$$A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 5 \end{bmatrix}$$

find the eigenvalues.

$$\det\left(A - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 7-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 3 & 5-\lambda \end{bmatrix}\right)$$

$$= (7-\lambda)(\lambda^2 - 6\lambda - 1) = 0$$

So either

$$7-\lambda=0 \rightarrow \lambda=7$$

or

$$\lambda^2-6\lambda-1=0 \rightarrow \lambda=3\pm\sqrt{10}$$

$$\boxed{\lambda = 7,\ 3\pm\sqrt{10}}$$

Example 4:

$$A = \begin{bmatrix} 56 & 11 & 73 \\ 64 & 10 & 49 \\ 13 & -56 & 7 \end{bmatrix}$$

Find the eigenvalues!

Run to Wolfram Alpha.

$\lambda \approx -24.6552$ is the only real eigenvalue.

Example 5: (reconstruction)

Suppose A is a 2×2 matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ corresponding to eigenvectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = x_1$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = x_2$. Find the matrix of A in the standard basis $\{e_1, e_2\}$.

We see that x_1 and x_2 are linearly independent and so form a basis for $\mathbb{R}^2$. We know

$$Ax_1 = x_1 \quad (\lambda_1 = 1)$$
$$Ax_2 = 2x_2 \quad (\lambda_2 = 2)$$

So in the basis $\{x_1, x_2\}$,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Let $S = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then the matrix of A with respect to $\{e_1, e_2\}$ is

$$\boxed{S \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} S^{-1}}$$